KLUB NAUKOWY
FENIX

Problems 7

6 November, 2020

Vectors

1. Express vector v as a linear combination of vectors $a, b, \ldots$ for following values of $v, a, \ldots$:

   (α) $v = [2,3], a = [1,-1], b = [1,1]$

   (β) $v = [1,1], a = [1,-1], b = [0,1]$

   (γ) $v = [1,1,1]$
   $a = [1,-1,0], b = [1,1,-2], c = [3,2,1]$

   (harder) $v = \cos^2(x), a = sin(2x), b = cos(2x), c = 1$

2. Find Euclidean length of following vectors

   (α) $[1,1,1]$

   (β) $[0,0,1]$

   (γ) $[\sin(x), \cos(x), 0]$

   (δ) $[\sin(x), \cos(x)\sin(y), \cos(x)\cos(y)]$

3. Let $f(a,b,c) = (a \times b).c$ for $a, b, c \in \mathbb{R}^3$ show that:

   (α) $f(a,b,c) = f(c,a,b) = f(b,c,a)$

   (β) $f(a,b,c) = -f(a,c,b)$

   deduce that $f(a,b,a) = 0$.

4. Find orthogonal basis for $\mathbb{R}^3$ that contains vector $[1,1,1]$.

Matrices

1. By considering action on standard basis of $\mathbb{R}^3$ find matrix representation of following linear transformations:

   (α) Reflection in xy plane.

   (β) 90° rotation around x axis.

   (γ) Reflection in xy plane followed by 90° rotation around x axis.

   (δ) 90° rotation around x axis followed by reflection in xy plane

2. In basis $b_1 = [1,1]^T, b_2 = [1,-1]^T$ matrix M has representation $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Write this matrix representation in standard basis.

3. Find determinant of following matrices

   (α) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

   (β) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

   (γ) $\begin{bmatrix} \sin(x) & \cos(x) \\ -\cos(x) & \sin(x) \end{bmatrix}$

   (δ) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

   (ε) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

   (ζ) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   (η) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$

   (θ) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

   (harder) $N \times N$ matrix with all entries equal to 1 (hint: eigenvalues).

Eigenvectors and eigenvalues

1. A symmetric matrix is called positive-definite if all it's eigenvalues are positive. Which of the following are positive definite:

(α) $\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ (β) $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ (γ) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (δ) $\begin{bmatrix} -1 & 9 \\ 9 & -1 \end{bmatrix}$

2. Find values of real parameter q for which the following are positive definite:

   (α) $\begin{bmatrix} 1 & 1 \\ 1 & q \end{bmatrix}$ (β) $\begin{bmatrix} q & 1 \\ 1 & q \end{bmatrix}$ (γ) $\begin{bmatrix} q & 1 \\ 1 & q^{-1} \end{bmatrix}$ (δ) $\begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}$

3. (harder) Show that any eigenvalue of real symmetric matrix is real.

Physics

1. Compute inertia tensor of a rigid tetrahedron made of massless rods and equal point masses at each vertex.