Test 1 solutions

12 December, 2020

Problem 2

If the pendulum is tilted from the equilibrium position by a small angle ϕ three, not balanced, forces will act upon it: the gravitational force, and the forces acting from the contracted or elongated springs. The total torque on the pendulum is therefore:

$$\Gamma = -mgL\sin\phi - k\left(\frac{1}{2}L\right)^2\sin\phi - k\left(\frac{3}{4}L\right)^2\sin\phi$$

where the first term is the torque from the gravity, the second is the torque from the spring that is at distance $L/2$ from the center of rotation and the thirst term is the torque from the spring that is at distance $3L/4$ from the center of rotation. The second newtons law of motion reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}M = \Gamma$$

where M is the angular momentum of the pendulum. In presented case $M = mL^2\dot{\phi}$, where $\dot{\phi} = \mathrm{d}\phi/\mathrm{d}t$. Finally the equation of motion is give as:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi + \left(\frac{g}{L} + \frac{13}{16}\frac{k}{m}\right)\sin\phi = 0$$

For small oscillations around the equilibrium position ($\phi = 0$) the $\sin\phi \approx \phi$, and the equation of motion can be simplified to:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi + \omega^2\phi = 0$$

where:

$$\omega = \sqrt{\frac{g}{L} + \frac{13}{16}\frac{k}{m}}$$

Finally the period of small osculations is:

$$T = \frac{2\pi}{\omega} = 2\pi\left(\frac{g}{L} + \frac{13}{16}\frac{k}{m}\right)^{-1/2}$$

Problem 3

In the linear AC circuits the relation between amplitude of the current J and the amplitude of the voltage U drop on the electric elements is given by the relation $U = |Z|J$, where the Z is the complex impedance of the circuit. For a resistor with resistance R the impedance is given as: $Z_R = R$, for an inductor with inductance L the impedance is given as: $Z_L = i\omega L$, and for a capacitor with capacitance C the impedance is given as: $Z_L = 1/(i\omega C)$. Here ω denotes the angular frequency of the applied voltage $U(t) = U\cos(\omega t)$. In the presented case the total impedance of the circuit is found to be:

$$Z = Z_R + Z_C + \frac{1}{\frac{1}{Z_R} + \frac{1}{Z_L}}$$

what simplifies to:

$$Z = R - \frac{i}{\omega C} + \frac{LR\omega}{L\omega - iR}$$

this summarised the first part of the task. The magnitude of the impedance $|Z| = \sqrt{ZZ^*}$ equals to:

$$|Z| = \sqrt{4R^2 + \frac{1}{C^2\omega^2} - \frac{2LR^2 + 3CR^2}{CR^2 + CL^2\omega^2}} \tag{1}$$

With a constant value of the U, the J will be the largest when $|Z|$ is the smallest. First lets check the value of (1) at the edges of interval, namely $\omega = 0$ and $\omega \to \infty$. If ω is large the second and the third term of (1) are small therefore the first term becomes dominant. In other words:

$$\lim_{\omega\to\infty} |Z(\omega)| = 2R$$

This is intuitive as in the presented case for large frequencies the impedance of the capacitor drops to zero and the impedance of the inductor rises to infinity. With similar arguments one can see that:

$$\lim_{\omega\to 0} |Z(\omega)| = \inf$$

Out of the two cases, current amplitude J will be largest for $\omega \to \inf$ and equal to $J_{\max} = U/(2R)$.

Now we can proceed to the discussion of possible function minima in the interval of $\omega \in (0, \infty)$. For simplicity, since square root is monotonic and increasing function, the value under the square root in the expression for $|Z|$ will be optimised. Function $|Z|^2$ is minimal when its derivative with respect to ω is zero:

$$\frac{\partial |Z|^2}{\partial \omega} = \frac{2}{C^2\omega^3}\left(\frac{CL^2\left(3CR^4 + 2LR^2\right)\omega^4}{\left(R^2 + L^2\omega^2\right)^2} - 1\right) = 0$$

which simplifies to a quadratic equation with respect to ω^2:

$$L^2\left(L + CR^2\right)\left(L - 3CR^2\right)\left(\omega^2\right)^2 + 2L^2R^2\omega^2 + R^4 = 0 \tag{2}$$

Study of the solutions to the equation is began with calculating its discriminant:

$$\Delta = 4CL^2R^6\left(3CR^2 + 2L\right)$$

since all values: L, R, C are positive the the discriminant positive and there are two solutions to the (2) equation.

Two roots to the equation (2) out of which one corresponds to the minimum and one to the maximum of the function (1) are given by:

$$\omega_1^2 = \frac{-R^4}{\sqrt{CL^2R^6\left(2L + 3CR^2\right)} + L^2R^2}$$
$$\omega_2^2 = \frac{R^4}{\sqrt{CL^2R^6\left(2L + 3CR^2\right)} - L^2R^2}$$

Above solutions are valid candidates for a minimum if they are positive (only real values of ω are considered). First solution (ω_1) is always negative and can be discarded at this point. Second solution (ω_2) is positive only if:

$$\sqrt{CL^2R^6\left(2L + 3CR^2\right)} > L^2R^2$$

since both sides of equation are positive one can square them and simplify the condition to:

$$L^2R^2\left(3CR^2 - L\right)\left(L + CR^2\right) > 0$$

which states that the ω_2^2 is positive if and only if $\left(3CR^2 - L\right) > 0$. It is left to determine if a minimum at ω_2 is a global minimum. The minimum of the function could possibly be taken on the edges of the interval. Those case have already been discussed above. The value of the minimum at $\omega = \omega_2$ has to be compared with the value $2R$ in order to determine which is lower:

$$|Z(\omega_2)|^2 = \frac{1}{C^2R^4}\left(-L^2R^2 + CR^4\left(CR^2 - 2L\right) + 2\sqrt{CL^2R^6\left(2L + 3CR^2\right)}\right) < 4R^2$$

which can be simplified to condition:

$$R^2\left(3CR^2 - L\right)^2\left(L + CR^2\right)^2 > 0$$

which for positive values of R, L, C is always true.

Summarising, the value of ω at which the current flowing through the circuit is maximal is achieved when the absolute value of the impedance of the circuit is minimal. The value of omega was found to be:

$$\omega_{\max} = \begin{cases} \infty & 3CR^2 - L \leq 0 \\ R^2\left(\sqrt{CL^2R^6\left(2L + 3CR^2\right)} - L^2R^2\right)^{-1/2} & 3CR^2 - L > 0 \end{cases}$$

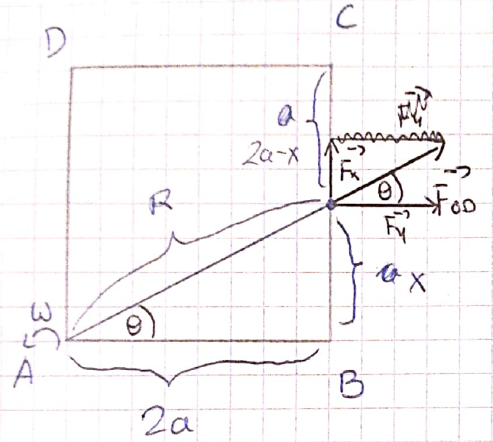
and the value of maximal current in the system is equal to:

$$J_{\max} = \begin{cases} \frac{U}{2R} & 3CR^2 - L \leq 0 \\ \frac{UCR^2}{\sqrt{-L^2R^2 + CR^4(CR^2 - 2L) + 2\sqrt{CL^2R^6(3CR^2 + 2L)}}} & 3CR^2 - L > 0 \end{cases}$$

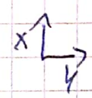
Zad 1 MACIEK DĄBKOWSKI

W zadaniu danej mamy kwadratową obręcz o boku o długości $2a$, która obraca się względem swojego wierzchołka A ze stałą prędkością kątową ω. Na środku boku BC obręczy znajduje się koralik, który może się po tej obręczy poruszać.

Na koralik działa pozioma siła odśrodkowa, która będzie powodowała ruch koralika po obręczy. Dodatkowo występuje jeszcze siła ciężkości, jednak nie ma ona wpływu na rozwiązanie, ponieważ jest równoważona przez siłę reakcji podłoża, a tarcia nie uwzględniamy.



Rysunek przedstawia rzut obracającej się ramki.



Siłę odśrodkową działającą na koralik możemy rozdzielić na składowe w osi x i y.

$$\vec{F_{OD}} = \vec{F_x} + \vec{F_y}$$

Zauważmy, że siła $\vec{F_y}$ będzie równoważona przez siłę reakcji pochodzącą od obręczy. Zatem jedyną siłą powodującą ruch koralika będzie siła $\vec{F_x}$!

Możemy zapisać, że:

$$F_{OD} = m\omega^2 R$$

$$F_x = \sin\theta \cdot F_{OD} = m\omega^2 R \sin\theta$$

Dodatkowo przyjmując x - odległość od wierzchołka B do koralika możemy zapisać, że:

$$R = \sqrt{4a^2+x^2}$$

$$\sin\theta = \frac{x}{R} = \frac{x}{\sqrt{4a^2+x^2}}$$

Stąd:

$$F_x = m\omega^2 \cdot \sqrt{4a^2+x^2} \cdot \frac{x}{\sqrt{4a^2+x^2}} = m\omega^2 x$$

Zatem z II Zasady Dynamiki Newtona

$$\ddot{x} m = m\omega^2 x$$

$$\ddot{x} - \omega^2 x = 0$$

Zależność ta jest pierwszą rzeczą, którą trzeba było w zadaniu pokazać. Teraz należy obliczyć czas jaki zajmie koralikowi dostanie się od położenia początkowego - środka boku BC do wierzchołka C.

Postulujemy rozwiązanie $x = A\exp(\alpha t)$

Wówczas $\ddot{x} = A\alpha^2 \exp(\alpha t)$

Zatem:

$$A\alpha^2 \exp(\alpha t) - \omega^2 A \exp(\alpha t) = 0$$

Rozwiązanie $\exp(\alpha t) = 0$ nas nie interesuje, tak samo jak rozw $A = 0$, zatem:

$$\alpha^2 - \omega^2 = 0$$

$$(\alpha - \omega)(\alpha + \omega) = 0$$

Stąd wynika, że $\alpha = \omega \quad \vee \quad \alpha = -\omega$

Otrzymujemy więc:

$$x_1 = A_1 \exp(\omega t) \qquad x_2 = A_2 \exp(-\omega t)$$

W szczególności interesuje nas rozwiązanie:

$$x = x_1 + x_2 = A_1 \exp(\omega t) + A_2 \exp(-\omega t)$$

Z warunków początkowych wiemy, że:

$$x(t=0) = a$$

$$\dot{x}(t=0) = 0$$

Zatem:

$$\begin{cases} A_1 \exp(0) + A_2 \exp(0) = a \\ A_1 \omega \exp(0) - A_2 \omega \exp(0) = 0 \end{cases}$$

$$\begin{cases} A_1 + A_2 = a \\ A_1 - A_2 = 0 \end{cases}$$

Rozwiązanie $\omega = 0$ nas nie interesuje

Stąd:

$$A_1 = A_2 = \frac{1}{2} a$$

Możemy więc zapisać:

$$x(t) = \frac{1}{2} a \exp(\omega t) + \frac{1}{2} a \exp(-\omega t)$$

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Gdy koralik znajduje się w rogu, to

$x(T) = 2a$, gdzie T – szukany czas, po którym koralik znajdzie się w rogu.

Zatem:

$$2a = \frac{1}{2} a \left(\exp(\omega T) + \exp(-\omega T) \right)$$

$$2 = \frac{1}{2} \left(\exp(\omega T) + \exp(-\omega T) \right)$$

Z definicji cosinusa hiperbolicznego wiemy, że:

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

Możemy zapisać więc, iż:

$$2 = \cosh(\omega T)$$

$$\operatorname{arccosh} 2 = \omega T$$

$$T = \frac{\operatorname{arccosh} 2}{\omega} = \cosh^{-1} 2 \cdot \omega^{-1}$$