Magnets

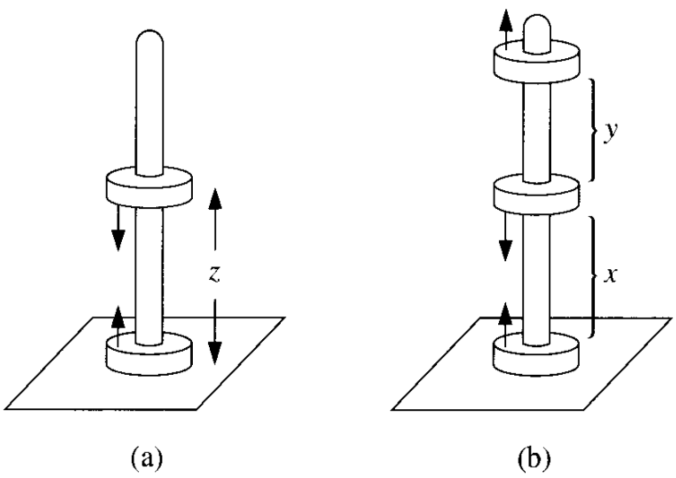
September 2023

Task

A familiar toy consists of donut-shaped permanent magnets (magnetization parallel to the axis), which slide frictionlessly on a vertical rod (Figure). Treat the magnets as dipoles, with mass m_d and dipole moment m.

(a) If you put two back-to-back magnets on the rod, the upper one will "float"—the magnetic force upward balancing the gravitational force downward. At what height (z) does it float?

(b) [HARD Problem] If you now add a third magnet (parallel to the bottom one), what is the ratio of the two heights? (Determine the actual number, to three significant digits.)



Source: David J. Griffiths, "Introduction to Electrodynamics".

Solution

(a) Forces acting on an upper magnet are:

$$\begin{aligned} \text{gravity force:} \quad & F_\mathrm{g} = -Mg \\ \text{magnetic interaction:} \quad & F_\mathrm{m} = \frac{3\mu_0}{2\pi}\frac{m^2}{z^4} \end{aligned} \tag{1}$$

Where M is the mass of magnets, g is the gravitational acceleration, μ_0 is the permeability of free space, m is the dipole moments of the magnets, and z is the distance between the magnets.

Minus sign for the gravity force reflects the fact that it acts in the opposing direction to the magnetic force. The net force is given as the sum of the two

$$F_\mathrm{net} = F_\mathrm{m} + F_\mathrm{g} = 0, \tag{2}$$

when stationary the net force equal zero. Substituting previous formulas, we get:

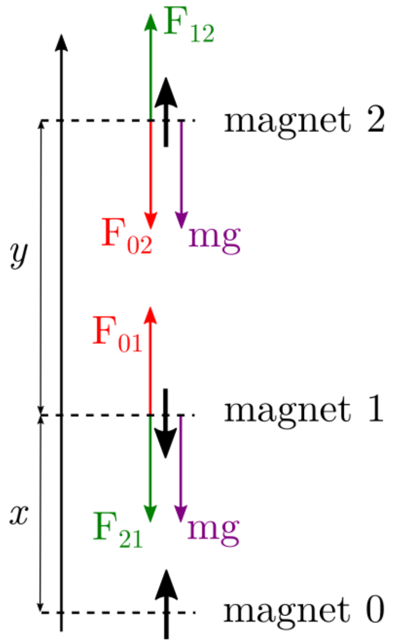
$$Mg = \frac{3\mu_0}{2\pi}\frac{m^2}{z^4}. \tag{3}$$

Now all that is left to do is to solve for z:

$$\begin{aligned} z^4 &= \frac{3\mu_0}{2\pi}\frac{m^2}{Mg} \\ z &= \left(\frac{3\mu_0}{2\pi}\frac{m^2}{Mg}\right)^{1/4} \end{aligned} \tag{4}$$

which is the solution for the distance between the floating magnets.

(b) The two magnets are floating in place, that means that the net force acting on each of them is zero. All of the forces are depicted on the drawing below:



All forces are assumed to be acting on the geometrical centres of magnets (there are drawn to the side for better visibility). The magnets 1 and 2 are repelling each other with forces F_{12} and F_{21} respectively. List of all the forces and their value is shown below.

upper magnet:

$$\begin{aligned} \text{gravity force:} &\quad F_g = -Mg \\ \text{magnetic interaction with resting magnet:} &\quad F_{02} = -\frac{3\mu_0}{2\pi}\frac{m^2}{(x+y)^4} \\ \text{magnetic interaction with lower magnet:} &\quad F_{12} = \frac{3\mu_0}{2\pi}\frac{m^2}{y^4} \end{aligned} \tag{5}$$

lower magnet:

$$\begin{aligned} \text{gravity force:} &\quad F_g = -Mg \\ \text{magnetic interaction with resting magnet:} &\quad F_{01} = \frac{3\mu_0}{2\pi}\frac{m^2}{x^4} \\ \text{magnetic interaction with lower magnet:} &\quad F_{21} = -\frac{3\mu_0}{2\pi}\frac{m^2}{y^4} \end{aligned}$$

by applying the no-net-force conditions we arrive with with a set of two equations:

$$\begin{aligned} \text{upper magnet:} \quad 0 &= -Mg - \frac{3\mu_0}{2\pi}\frac{m^2}{(x+y)^4} + \frac{3\mu_0}{2\pi}\frac{m^2}{y^4} \\ \text{lower magnet:} \quad 0 &= -Mg + \frac{3\mu_0}{2\pi}\frac{m^2}{x^4} - \frac{3\mu_0}{2\pi}\frac{m^2}{y^4} \end{aligned} \tag{6}$$

Here the *physics* ends and the rest is just algebra. Simplifying a bit:

$$\begin{aligned} Mg &= \frac{3\mu_0 m^2}{2\pi}\left(-\frac{1}{(x+y)^4} + \frac{1}{y^4}\right) \\ Mg &= \frac{3\mu_0 m^2}{2\pi}\left(\frac{1}{x^4} - \frac{1}{y^4}\right) \end{aligned} \tag{7}$$

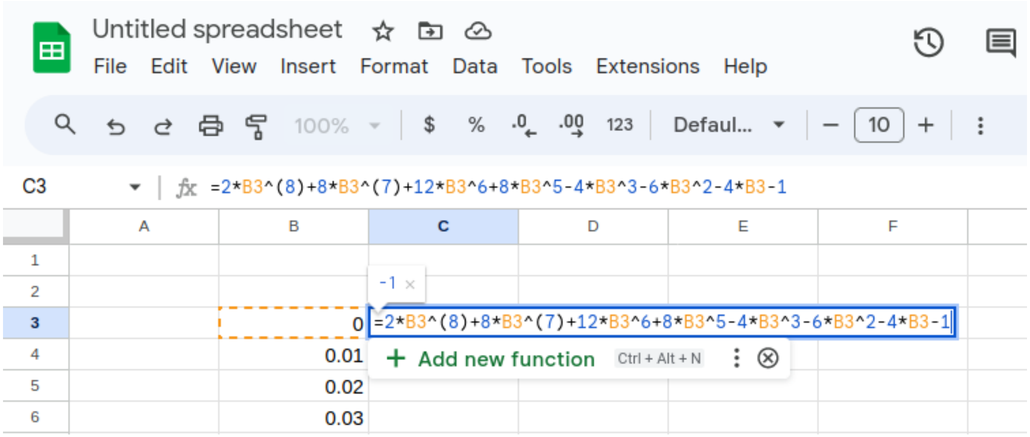
sing the left side of the two equations are equal, so are the right sided, combining the two equations and canceling out term $\frac{3\mu_0 m^2}{2\pi}$ results in:

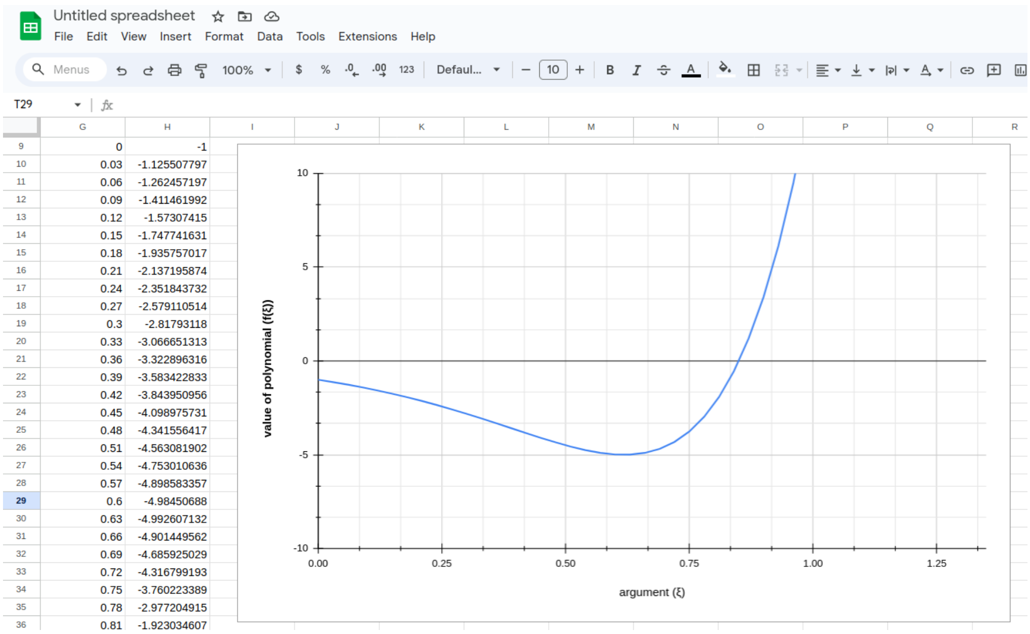
$$\begin{gathered} -\frac{1}{(x+y)^4} + \frac{1}{y^4} = \frac{1}{x^4} - \frac{1}{y^4} \\ \text{multiplyingby } y^4 \text{ gives :} \\ -\frac{y^4}{(x+y)^4} + 1 = \frac{y^4}{x^4} - 1 \\ -\left(\frac{1}{\frac{x}{y}+1}\right)^4 + 1 = \left(\frac{y}{x}\right)^4 - 1 \end{gathered} \tag{8}$$

In the end the equation can be solved for a single variable $\xi = x/y$:

$$\begin{gathered} -\left(\frac{1}{\xi+1}\right)^4 + 1 = \frac{1}{\xi^4} - 1 \\ -\frac{1}{\xi^4 + 4\xi^3 + 6\xi^2 + 4\xi + 1} + 2 - \frac{1}{\xi^4} = 0 \\ \text{multiplying both by } \xi^4(\xi^4 + 4\xi^3 + 6\xi^2 + 4\xi + 1): \\ 2\xi^8 + 8\xi^7 + 12\xi^6 + 8\xi^5 - 4\xi^3 - 6\xi^2 - 4\xi - 1 = 0 \end{gathered} \tag{9}$$

results in the final equation for ξ. To solve this challenging equation, we'll use a numerical methods to find an approximate solution. We'll utilize Excel to help us with this. By plotting the polynomial in Excel, we can visually examine where it equals 0.





As seen from the plot, the area of interest of the value ξ is somewhere between 0.8 and 0.9. We can refine out search by increasing the precision of the computations up to a third decimal place, and find the value for which the polynomial value is closest to zero.

O	P	Q	R	S	T	U
0.847	0.848	0.849	0.85	0.851	0.852	0.853
-0.1730931738	-0.1179917876	-0.06245003761	-0.00646555617	0.04996403406	0.1068411203	0.1641680996

We found it to be $\xi \approx 0.850$. More sophisticated methods can be used to achieve even higher precision. For example one can use Wolfram Mathematica software:

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NSolve[2 ξ^8 + 8 ξ^7 + 12 ξ^6 + 8 ξ^5 - 4 ξ^3 - 6 ξ^2 - 4 ξ - 1 == 0, ξ, NonNegativeReals]

{{ξ → 0.850115}}
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Which is an identical result as we achieved using excel.