Pendulums

6th of October 2023

Tasks

1. The rotational motion in a single plane can be described using the equation: $\Gamma = I\varepsilon$. Find the units of Γ,I,ε.

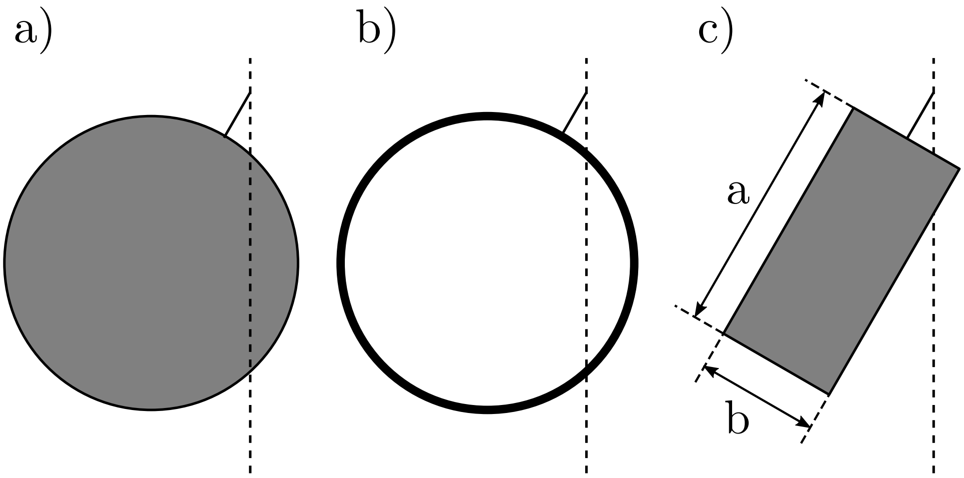
2. Derive the formula for the periods, and calculate their values for the following pendulums (in each case, the centre of mass is at the distance $L = 1\,\mathrm{m}$ from the centre of rotation and the mass of the objects is $M = 1\,\mathrm{kg}$):

   a) pendulum with a disk with radius $R = 0.9\,\mathrm{m}$;

   b) pendulum with a circular loop with radius $R = 0.9\,\mathrm{m}$;

   c) pendulum with a rectangular plate with edge lengths of $a = 1\,\mathrm{m}$, $b = 0.5\,\mathrm{m}$;

   You can use Wikipedia (List of moments of inertia) for reference.



3. [HARD Problem] Derive the formula for the motion of a pendulum, without making the small-angle approximation. Calculate the period of such a pendulum and plot its dependency on the initial angle. You can assume that the mass moment of inertia is $I = mL^2$ and the distance from the axis of rotation to the centre of mass is $L = 1\,\mathrm{m}$.

Solution

1. The general form of the formula for I can be expressed as:

$$I = \beta m r^2 \tag{1}$$

where m is mass [kg], r is characteristic length [m], and β is a dimensionless numeric constant. The usnits for I can be computes as:

$$[I] = [m][r]^2 = \text{kg m}^2 \tag{2}$$

Torque Γ can be calculated as force times the arm of the force r:

$$\Gamma = Fr \tag{3}$$

computing the units we get:

$$[\Gamma] = \text{Nm} = \frac{\text{kg m}^2}{\text{s}^2} \tag{4}$$

With these we can proceed to determining units of ε as:

$$[\varepsilon] = \left[\frac{\Gamma}{I}\right] = \frac{\text{kg m}^2}{\text{s}^2}/\text{kg m}^2 = \frac{1}{\text{s}^2} \tag{5}$$

2. We will denote the moment of inertia around the centre of mass ad I_{CM}. It is important to distinguish I_{CM} from the moment of inertia around the rotational axis I, which does not necessarily pass through the center of mass. Relation between the two can be found using the Steiner's formula:

$$I = I_{CM} + ML^2 \tag{6}$$

Knowing the moment of inertia around the rotational axis standard form of the equation of motion can be used:

$$\Gamma = I\varepsilon \tag{7}$$

Substituting the formula for the moment of inertia and the torque:

$$-MgL\sin(\varphi) = \left(I_{CM} + ML^2\right)\varepsilon \tag{8}$$

simplifying a bit and assuming small oscillations we get:

$$-\frac{MgL}{I_{CM} + ML^2}\varphi = \varepsilon \tag{9}$$

This equation can be recognised as the equation for a simple harmonic pendulum. In that case the frequency is:

$$\omega^2 = \frac{MgL}{I_{CM} + ML^2} \tag{10}$$

and the period is:

$$T = 2\pi\sqrt{\frac{I_{CM} + ML^2}{MgL}} \tag{11}$$

Now we can proceed to solve each case.

a) The moment of inertia around the center of mass for a filled disk is (from Wikipedia):

$$I_{CM} = \frac{1}{2}MR^2 \tag{12}$$

giving:

$$T = 2\pi\sqrt{\frac{R^2 + 2L^2}{2gL}} \tag{13}$$

Substituting the given values:

$$T = 2.378\,\mathrm{s} \tag{14}$$

b) The moment of inertia around the center of mass for a circle is (from Wikipedia):

$$I_{CM} = MR^2 \tag{15}$$

giving:

$$T = 2\pi\sqrt{\frac{R^2 + L^2}{gL}} \tag{16}$$

Substituting the given values:

$$T = 2.699\,\mathrm{s} \tag{17}$$

c) The moment of inertia around the center of mass for a rectangle is (from Wikipedia):

$$I_{CM} = \frac{1}{12}M(a^2 + b^2) \tag{18}$$

Substituting the given values:

$$T = 2.108\,\mathrm{s} \tag{19}$$

3. Without the small angle approximation the equation of motion is:

$$-\frac{g}{l}\sin(\varphi) = \ddot{\varphi} \tag{20}$$

This equation is hard to solve using standard methods, we will proceed with numerical analysis. Let's analyse equation of motion considering small time steps, of length dt. Within each step we will assume that the acceleration $\ddot{\varphi}$ is *constant*. For each such time step the solution for position at time $t + dt$ can be calculated as:

$$\varphi(t + \mathrm{d}t) = \varphi(t) + \dot{\varphi}\mathrm{d}t + \frac{1}{2}\ddot{\varphi}\mathrm{d}t^2 \tag{21}$$

where $\varphi(t)$ is the position at time t, ω is angular velocity at time t, and $\ddot{\varphi}$ is the acceleration at time t. Additionally

$$\omega(t + \mathrm{d}t) = \dot{\varphi} + \ddot{\varphi}\mathrm{d}t \tag{22}$$

We will use the above equations to compute $\ddot{\varphi}$, $\dot{\varphi}$ and φ for each time step, starting from initial values of $\varphi = \varphi_0$ and $\dot{\varphi} = 0$. In excel first we define a column with time of each time step:

1	time		
2	0		
3	0.005		
4	0.01		
5	0.015		
6	0.02		

where time starts at 0 s, ends at 2 s with a time step dt of 0.005 s. With initial values $\varphi(0) = \varphi_0$ and $\dot{\varphi} = 0$ and write formula for acceleration:

	A	B	C	D
1			constant:	-9.81
2	initial angle	0.01		
3	time	angle	velocity	acceleration
4	0	0.01		`=$D$1*sin(B4)`
5	0.005			0

Notice that the cell containing the value of constant $-g/l$ is fixed by using \$ sign. Here we wrote acceleration formula for all values of φ. Now we can proceed with writing formula for angle:

B5 `=B4+C4*0.005+(1/2)*D4*0.005*0.005`

	A	B	C	D
1			constant:	-9.81
2	initial angle	0.01		
3	time	angle	velocity	acceleration
4	0	0.01	0	-0.09809836501
5	0.005	`=B4+C4*0.005+(1/2)*D4*0.005*0.005`		30863363

and angular velocity:

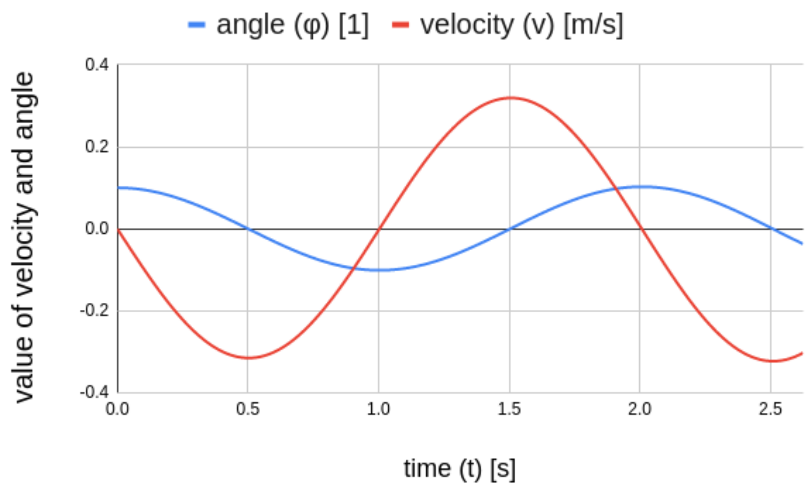
C5 `=C4+D4*0.005`

	A	B	C	D
1			constant:	-9.81
2	initial angle	0.01		
3	time	angle	velocity	acceleration
4	0	0.01	0	-0.09809836501
5	0.005	0.00999877377	`=C4+D4*0.005`	-0.0980863363

Now we just pull formula boxes down in order to calculate next steps:

2	initial angle	0.01		
3	time	angle	velocity	acceleration
4	0	0.01	0	-0.09809836501
5	0.005	0.00999877377	-0.00049049182	-0.0980863363
6	0.01	0.009995095232	-0.00098092350	-0.09805025164
7	0.015	0.009988964986	-0.00147117476	-0.09799011693
8	0.02	0.009980384236	-0.00196112534	-0.09790594397

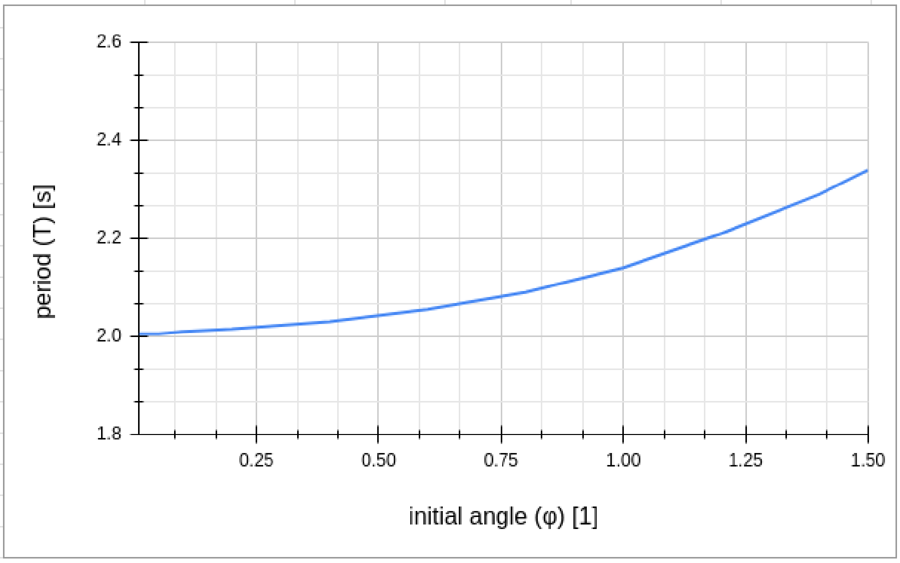
We can see that value of angle is decreasing(pendulum is falling) and velocity is increasing. After solving more in time domain we can plot solution.



And then trace in raw data when solution passes whole period.

Doing so for different initial angles we get a solution to problem:

initial angle	period
0.01	2.005
0.05	2.005
0.1	2.01
0.2	2.015
0.4	2.03
0.6	2.055
0.8	2.09
1	2.14
1.2	2.21
1.4	2.29
1.5	2.34



As mentioned before, solving the nonlinear equation is hard but not impossible! Since the energy of the system is conserved:

$$-mgl\cos(\varphi_0) = \frac{ml^2\dot{\varphi}^2}{2} - mgl\cos(\varphi) \tag{23}$$

Where left hand side is a total energy at the beginning of the motion and the right hand side is energy at any moment of motion. Solving gives:

$$\dot{\varphi} = \left(\sqrt{\frac{2g}{l}}\right)\sqrt{\cos(\varphi) - \cos(\varphi_0)} \tag{24}$$

Now one needs to recall fact that $\dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}$. Let's call $\xi = \sqrt{\frac{2g}{l}}$ and rewrite our formula:

$$\mathrm{d}t = \frac{\mathrm{d}\varphi}{\xi\sqrt{\cos(\varphi) - \cos(\varphi_0)}} \tag{25}$$

The physical meaning of above formula is as follows: traveling distance $\mathrm{d}\varphi$ takes different amount of time for different angles. For example at $\varphi \approx \varphi_0$ it takes much longer then for $\varphi = 0$ hence angular velocity is very small. Now we need to sum times to get how much it will take to travel from $\varphi = \varphi_0$ to $\varphi = 0$ and it is exactly $1/4$ of a period. Such sum over continuous very small distances is in fact an integral:

$$\frac{T}{4} = \frac{1}{\xi}\int_{\varphi_0}^{0} \frac{\mathrm{d}\varphi}{\sqrt{\cos(\varphi) - \cos(\varphi_0)}} \tag{26}$$

This is an elliptic integral. Such functions cannot be expressed using standard functions, but its values have been well studied and tabulated in great detail. Using the below scrip in Mathematica, we've solved integral using function Integrate[] and solution is as we expected dependent on tabulated function EllipticF[a,b]. It takes arguments and returns a value of this elliptic integral from table of values. Then using solution we calculated values of period for different φ_0 and assuming $g = 9.81\,\mathrm{m/s^2}$. As the function may be not intuitive it is shown on the plot below.

```
In[40]:= int = Integrate[1/Sqrt[Cos[φ] - Cos[φ0]], {φ, φ0, 0}]

Out[40]=
-(2 EllipticF[φ0/2, Csc[φ0/2]^2])/Sqrt[1 - Cos[φ0]]

In[41]:= Table[4/ξ int, {φ0, 0, π/2, 0.01}]
```